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Algebra.

Q.1. The relation of isomorphism in the set of all groups is an equivalence relation.

Proof: Let G, G', G'' be three groups. Then

(I) Reflexive. For this let us consider the map $f: G \to G$ given by $f(x) = x$, $\forall x \in G$. Evidently f is one-one onto.

Also, $f(xy) = xy$, $\forall x, y \in G$
$= f(x) f(y)$

$\therefore f$ preserves the composition in G.
$\Rightarrow G \cong G$.
$\Rightarrow \cong$ is Reflexive.

(II) Symmetric. $G \cong G' \Rightarrow G' \cong G$

For, $G \cong G' \Rightarrow$ there exists an isomorphism $f: G \xrightarrow{onto} G'$
$\Rightarrow f$ is one-one onto and $f(xy) = f(x) f(y)$, $x, y \in G$.
$\Rightarrow f^{-1}: G' \to G$ is one-one onto.

Let $x', y' \in G'$, then there exists $x, y \in G'$ such that
$$f^{-1}(x') = x, \; f^{-1}(y') = y$$

Then, $f(x) = x'$, $f(y) = y'$

$f^{-1}(x'y') = f^{-1}[f(x) f(y)] = f^{-1}[f(xy)] = (f^{-1}f)(xy)$
$= xy$ [$\because f^{-1}f$ is an identity map]
$= f^{-1}(x') f^{-1}(y')$

Finally, $f^{-1}(x'y') = f^{-1}(x') f^{-1}(y')$

$\Rightarrow f^{-1}$ preserves compositions in G and G'.

Also, we have that f^{-1} is one-one onto.

$\therefore f^{-1}: G' \to G$ is an isomorphism
$\Rightarrow G' \cong G$
$\cong$ is symmetric.

(III) Transitive: $G \cong G'$, $G' \cong G'' \Rightarrow G \cong G''$

For, $G \cong G'$. $G' \cong G''$ There exists isomorphism $f: G \xrightarrow{onto} G'$, $g: G' \to G''$. $\Rightarrow f$ and g are one-one onto maps and $f(xy) = f(x) f(y)$, $x, y \in G$ $\therefore g(x'y') = g(x') g(y')$: $x', y' \in G'$.

$\Rightarrow gf: G \to G''$ is one-one onto.

And $(gf)(xy) = g[f(xy)]$; $x, y \in G$.
$= g[f(x) f(y)] = g[f(x)] g[f(y)]$
$= (gf)(x) (gf)(y)$, i.e. gf is order preserving.

$\Rightarrow$ If $f: G \to G''$ is an isomorphism.

$\Rightarrow G \cong G''$.

$\Rightarrow \cong$ is transitive.

Thus, the relation satisfies all the conditions of an equivalence relation and hence the given relation is an equivalence relation.

Q.2. Show that the additive group of integers $G = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$ is isomorphic to the additive group $G' = \{\ldots -3m, -2m, -1m, 0, 1m, 2m, 3m, \ldots\}$ where m is any fixed integer not equal to zero.

Soln: Let $a \in G$. Then, we have $ma \in G'$.

Define a map $f: G \to G'$ such that $f(a) = ma \quad \forall a \in G$.

(I) f is one-one. Let $a_1, a_2 \in G$. Then $f(a_1) = f(a_2)$

$\Rightarrow ma_1 = ma_2$ [By definition of f]

$\Rightarrow a_1 = a_2$

$\Rightarrow f$ is one-one.

II f is onto. Since, m is any fixed integer. Therefore, for one $a \in G$ there exists only one element ma in G', which implies that G and G' both have the same number of elements.

Also, f is one-one.

Hence, f is onto.

III Structure preserving property:

Let $a_1, a_2 \in G$ then

$$f(a_1 + a_2) = m(a_1 + a_2) = ma_1 + ma_2 = f(a_1) + f(a_2)$$

$\Rightarrow f$ preserves composition in G and G'.

Finally, from I, II and III we conclude that f is an isomorphism.

$\Rightarrow G \cong G'$.